Part VII

- Physics of Ideal Fermi Gas
  [Mostly 3D, non-relativistic, free fermions]

- Physics of Ideal Bose Gas
  [Mostly Bose-Einstein Condensation]

XIII. Ideal Fermi Gas

Model System: A non-interacting gas of fermions inside a large 3D box of volume V

- 3D
- Particle-in-a-big-Volume
- Non-relatistic
- dispersion relation $\varepsilon = \frac{\hbar^2 k^2}{2m}$

Possible real systems

- gas of conduction electrons in a metal — 10^{22-23} electrons per cm^3
- ^{3}He liquid
- (With modifications) fermions in neutron stars/white dwarf

Key information reflecting system properties

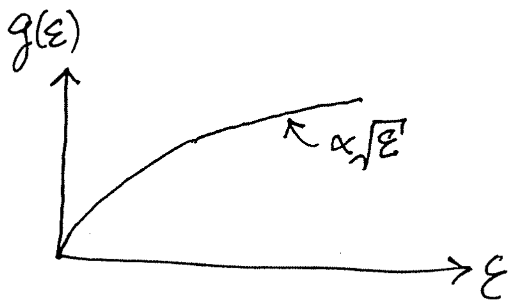
$$g(\varepsilon) = \underbrace{G_s}_{\text{spin-degeneracy}} \cdot \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$$

density of (single-particle) states

[Typically, spin-half $\Rightarrow G_s = 2s+1 = 2\cdot\frac{1}{2}+1 = 2$]

Thus

$$g(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$$



Note: $g(\varepsilon) \propto \varepsilon^{1/2}$

$g(\varepsilon) \propto V$

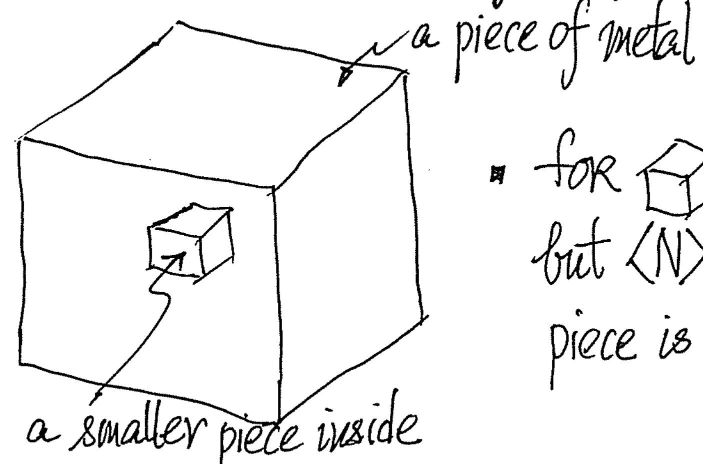
Key ideas in applying Stat. Mech. formulas

- A piece of metal $\frac{N}{V} = \frac{\#\text{conduction electrons}}{\text{Volume}}$ = electron number density is a property of the material

$\frac{N}{V} \equiv n$, $n(Ag) \neq n(Cu) \neq n(Na) \neq n(Au)$

But $\underbrace{n(Ag)}_{\text{smaller piece}} = \underbrace{n(Ag)}_{\text{bigger piece}}$ (same material)

- For metals (solids), they expand only a tiny bit between $T \sim 0K$ to $T \sim 300\text{-}400K$ that we can take $\frac{N}{V}$ (electron number density) as a material's property independent of temperature.
- In establishing the formulations, we used N and $\langle N \rangle$. However, $\langle N \rangle$ is highly representative of the particle number.



- for [cube] inside, N may fluctuate but $\langle N \rangle$ is sharp and $\frac{\langle N \rangle}{V}$ for small piece is the same as $\frac{\langle N \rangle}{V}$ for big piece.

- With this is mind, we will use N instead of $\langle N \rangle$ in the formulaes established in Ch. XII.
- Of course, this will not alter the physics.
- To have a sample system in mind when we go through the many equations, let's take conduction electrons in metals as our 3D Ideal Fermi Gas. It is useful to recall some facts in metal physics.
- Similarly, we will use E instead of $\langle E \rangle$, as they play the same role in thermodynamics.

To put the many equations that follow into a proper perspective, here are some facts in metal physics.

- Fermi Energy (in eV)

Li	Na	K	Cu	Au	Mg	Zn	Al
3.9	2.8	1.9	6.5	5.4	7.6	11.5	11.8

[From J.D. Livingston, "Electronic Properties of Engineering Materials"]

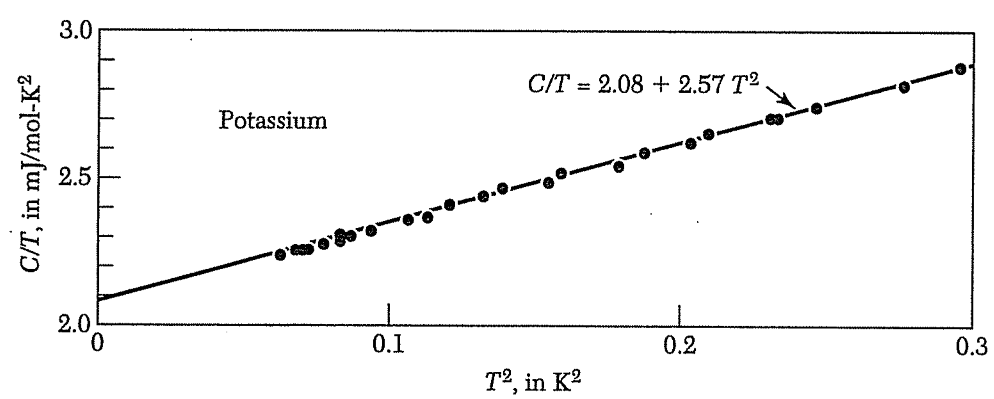
- Heat Capacity of Metals

$$C = \underbrace{\gamma T}_{??} + \underbrace{bT^3}_{\text{recall: Debye model gives} \sim T^3} \quad \text{(low temperatures)}$$

?? (due to sea of conduction electrons) [ideal Fermi Gas]

recall: Debye model gives $\sim T^3$ term due to vibrations of ions

$$\frac{C}{T} = \gamma + bT^2$$



[From C. Kittel, "Introduction to Solid State Physics"]

A. Equations for N, E, Ω, and Equation of State

$$g(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2} \qquad (1) \qquad [G_s = 2, \text{ spin-half assumed}]$$

$$N = \sum_{\text{all s.p. states } i} \underbrace{\frac{1}{e^{\beta(\varepsilon_i-\mu)}+1}}_{\text{\# fermion per single-particle state of energy } \varepsilon_i \text{ (meaning of } f_{FD}(\varepsilon_i))} \qquad (2) \; [\text{General}]$$

becomes

$$N = \int_0^\infty g(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon \qquad (2a) \; [\text{General}]$$

and further becomes

$$\boxed{N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon \qquad (3)}^{+}$$

3D non-relativistic in a Volume V

an equation for determining $\mu(T)$, given a material (given $\frac{N}{V}$).

$^+$Remark:

We started to see that we need to handle integrals of the form $\displaystyle\int_0^\infty \frac{f(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon$ ← some function of ε

$$E = \sum_{\text{all s.p. states } i} \varepsilon_i \cdot \frac{1}{e^{\beta(\varepsilon_i-\mu)}+1} \qquad (4) \; [\text{General}]$$

$$= \int_0^\infty g(\varepsilon) \cdot \varepsilon \cdot \frac{1}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon \qquad (4a) \; [\text{General}]$$

becomes

$$\boxed{E = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon \qquad (5)}$$

3D non-relativistic in a volume V

With $\mu(T)$ from Eq.(3), Eq.(5) gives $E(T)$. A derivative will give the heat capacity $C_V(T)$.

Grand Potential: $\Omega = -kT \ln \mathcal{Q}$

$$\Omega = -kT \sum_{\text{all s.p. states } i} \ln\left(1 + e^{-\beta(\varepsilon_i-\mu)}\right) \qquad (6) \; [\text{General}]$$

$$= -kT \int_0^\infty g(\varepsilon) \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right) d\varepsilon \qquad (6a) \; [\text{General}]$$

becomes

$$\boxed{\Omega = -kT \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \varepsilon^{1/2} \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right) d\varepsilon \qquad (7)}$$

3D non-relativistic in a volume V

Equation of State

Recall $\Omega = -pV$

$\therefore \quad pV = kT \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \underbrace{\int_0^\infty \varepsilon^{1/2} \ln(1+e^{-\beta(\varepsilon-\mu)})\,d\varepsilon}_{\text{do this integral by parts}}$

$= kT \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \ln(1+e^{-\beta(\varepsilon-\mu)})\left(\frac{d}{d\varepsilon}\varepsilon^{3/2}\right) d\varepsilon \underbrace{\left(\frac{2}{3}\right)}$

note how this factor comes about!

$= -\frac{2}{3} kT \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \varepsilon^{3/2}\left(\frac{d}{d\varepsilon}\ln(1+e^{-\beta(\varepsilon-\mu)})\right) d\varepsilon$

[note: the "surface term" vanishes]

$= \underbrace{\frac{2}{3} \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon}$

$= \frac{2}{3} E$

$\therefore \quad \boxed{pV = \frac{2}{3}E} \quad (8)$ for 3D non-relativistic free fermions in volume V [true for all temperatures!]

- How "$\frac{2}{3}$" comes about?
- Same steps also lead to $pV = \frac{2}{3}E$ for ideal Bose gas under same conditions
- Contrast Eq.(8) with photon gas.
- Eq.(8) $\Rightarrow \Omega = -\frac{2}{3}E$

- Eqs.(1) for DOS and Eqs.(3), (5), (8) can be applied to study the physics of 3D ideal Fermi Gas at different temperatures
- We also focus on
  - $T=0$ physics (dominating because of Pauli Principle)
  - Low-temperature physics (usually the case)
  - High-temperature correction to classical gas behavior

(see figure on following page)

We will study...

<u>T=0 physics</u>
(Pauli Exclusion Principle)

Very high temperature
(classical gas limit)

0 $\quad$ T_F $\quad$ temperature

$T \ll T_F$
(OR $kT \ll E_F$)
<u>low-temperature physics</u>
(degenerate Fermi Gas)

Fermi temperature
(set by system properties)

<u>High-temperature</u>
with correction to
classical ideal gas behavior